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ON THE ERROR PROPAGATION OF SOME INTERPOLATION FORMULAS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

By Erwin Fehlberg

George C. Marshall Space Flight Center
Huntsville, Alabama

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SUMMARY

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Simple extrapolation formulas are presented for the propagation of the truncation error for interpolation formulas integrating $y'' = f(x,y)$. These error formulas are applied to the interpolation formulas of Milne and Gauss, and their accuracy is illustrated by examples.

The error formulas are extended to cover the case of interpolation formulas for $y'' = f(x,y,y')$. The interpolation formulas of Milne and Gauss are supplemented by new formulas for y' in such a way that the combined formulas have the same small error parameter E as the original formulas of Milne or Gauss. Examples demonstrate the accuracy of the error formulas for these new interpolation formulas.

SECTION I. INTRODUCTION

This technical note deals with the propagation of the truncation error of interpolation formulas as used in the numerical integration of second-order differential equations.

Of primary interest are formulas that have a small truncation error or formulas that propagate only a small fraction of their truncation error. In an earlier paper[a] we investigated the error propagation of such interpolation formulas and presented a simple

a. E. Fehlberg, Numerically Stable Interpolation Formulas with Favorable Error Propagation for First- and Second-Order Differential Equations, NASA Technical Note D-599, March 1961.

formula for the ratio of the propagated truncation errors of two such interpolation formulas. In this note we investigate the propagated truncation error itself. We shall show how this error can be computed in conjunction with the computation of the solution of the differential equation.

The ratio of the propagated truncation errors is of interest in comparing different interpolation formulas. The propagated truncation error itself, if computed in conjunction with the solution of the differential equations, offers a convenient check of the accuracy of the computation.

We have simplified the theory of the propagated truncation error as much as possible, so that the necessary computations can be performed quickly and easily. Nevertheless, our examples do show that good results can be obtained, even for a large number of integration steps.

This paper does not deal with the round-off error and its propagation. We programmed our interpolation formulas for the digital computer in such fashion that the round-off error was reduced as much as possible. Then, to study the propagation of the truncation error, we selected an integration step size causing a dominant truncation error.

One important consideration in problems that require an extensive number of integration steps is that of reducing time on the computer. This can be done by increasing the integration step size, thereby committing a dominant truncation error. In such cases, however, it is almost mandatory to know the magnitude of the propagated truncation error in order to insure that the result of the integration is still sufficiently accurate.

Assume that the output of the digital computer consists of numbers with q digits. Then, in double precision, the computer will furnish numbers with 2q digits. If, after an extensive number of integration steps, results with q-digit accuracy are desired, the computation will have to be started with more than q correct digits (because of unavoidable error accumulation). In many cases, however, the computation will not have to be started with the maximum of 2q correct digits. Instead, it might suffice to select the integration step size in such fashion that the last q' digits ($0 < q' < q$) of the 2q-digit numbers are already affected by the truncation error. We then encounter a situation where the truncation error exceeds the round-off error. In such cases the propagated truncation error, computation of which is described in this technical note, is practically identical with the total error.

SECTION II. ERROR PROPAGATION OF INTERPOLATION FORMULAS FOR THE DIFFERENTIAL EQUATION $y'' = f(x,y)$

A. THEORY OF THE ERROR PROPAGATION

In this section we consider differential equations:

$$y'' = f(x,y) \tag{1}$$

and interpolation formulas of the form:

$$y_{k+1} = \sum_{\mu=0}^{m} B_{k-\mu}\, y_{k-\mu} + h^2 \sum_{\mu=-m'}^{m} B''_{k-\mu}\, f_{k-\mu} \tag{2}$$

The constants $B_{k-\mu}$ and $B''_{k-\mu}$ are determined by the Taylor expansion of (2) for $x = x_k$. Comparing the coefficients of the expansion for the first powers of h, the following equations are obtained ($B_{k+1} = -1$):

$$\left.\begin{aligned} &\sum_{\mu=-1}^{m} B_{k-\mu} = 0 \\ &\sum_{\mu=-1}^{m} \mu\, B_{k-\mu} = 0 \\ &\frac{1}{2} \sum_{\mu=-1}^{m} \mu^2 B_{k-\mu} + \sum_{\mu=-m'}^{m} B''_{k-\mu} = 0 \end{aligned}\right\} \tag{3}$$

If we denote the true values of the solution of (1) by $y(x_{k-\mu})$ and the approximate values of this solution as obtained from (2) by $y_{k-\mu}$, the errors:

$$y_{k-\mu} - y(x_{k-\mu}) = \epsilon_{k-\mu} \tag{4}$$

satisfy the equation:

$$\sum_{\mu=-1}^{m} B_{k-\mu}\, \epsilon_{k-\mu} + h^2 \sum_{\mu=-m'}^{m} B''_{k-\mu} \left(\frac{\partial f}{\partial y}\right)_{k-\mu} \epsilon_{k-\mu} + \dots = T_k \tag{5}$$

Here, T_k stands for the truncation error at $x = x_k$. The dots on the lefthand side of (5) are to indicate that we have omitted terms with $\epsilon^2_{k-\mu}$ and higher powers of $\epsilon_{k-\mu}$.

Such terms will henceforth be neglected. Let the interpolation formula (2) approximate the solution of (1) correctly up to the term h^p, inclusively, of the Taylor expansion. The first term of the truncation error T_k in (5) then reads:

$$T_k = C_{p+1} \, y^{(p+1)}(x_k) \, h^{p+1} \tag{6}$$

C_{p+1} being a constant that depends on $B_{k-\mu}$ and $B''_{k-\mu}$.

Inserting the Taylor expansion:

$$\epsilon_{k-\mu} = \epsilon_k - \mu h \, \epsilon'_k + \frac{1}{2} \mu^2 h^2 \, \epsilon''_k - \ldots + \ldots \tag{7}$$

in (5), neglecting all terms with h^3 and higher powers of h on the lefthand side of (5), and taking into account (3), we obtain:

$$\frac{1}{2} h^2 \, \epsilon''_k \sum_{\mu=-1}^{m} \mu^2 B_{k-\mu} - \frac{1}{2} h^2 \left(\frac{\partial f}{\partial y}\right)_k \epsilon_k \sum_{\mu=-1}^{m} \mu^2 B_{k-\mu} = C_{p+1} \, y^{(p+1)}(x_k) \, h^{p+1},$$

if only the first term of the truncation error is considered.

The last equation yields the following differential equation for the propagation of the truncation error:

$$\epsilon'' - \frac{\partial f}{\partial y} \epsilon = \frac{2 \, C_{p+1}}{\sum_{\mu=-1}^{m} \mu^2 B_{k-\mu}} \, y^{(p+1)}(x) \, h^{p-1} \tag{8}$$

By integrating (8), the propagated truncation error $\epsilon(x)$ can be obtained.

Let $\epsilon_1(x)$, $\epsilon_2(x)$ be a fundamental system of the homogeneous equation (8). Then the general solution of the non-homogeneous equation (8) can be written as:

$$\left.\begin{aligned}\epsilon(x) = \epsilon_1(x)&\left\{-\frac{2\,C_{p+1}}{\sum_{\mu=-1}^{m} \mu^2\, B_{k-\mu}}\, h^{p-1} \int_{x_0}^{x} \frac{\overset{(p+1)}{y}(x)\ \epsilon_2(x)}{\epsilon_1(x)\ \epsilon_2'(x) - \epsilon_1'(x)\ \epsilon_2(x)}\, dx + h_1\right\}\\ + \epsilon_2(x)&\left\{\frac{2\,C_{p+1}}{\sum_{\mu=-1}^{m} \mu^2\, B_{k-\mu}}\, h^{p-1} \int_{x_0}^{x} \frac{\overset{(p+1)}{y}(x)\ \epsilon_1(x)}{\epsilon_1(x)\ \epsilon_2'(x) - \epsilon_1'(x)\ \epsilon_2(x)}\, dx + h_2\right\}\end{aligned}\right\} \quad (9)$$

with h_1 and h_2 as constants of integration.

Let us assume that, for the initial point $x = x_0$, the initial values $y(x_0)$ and $y'(x_0)$ have no error. This means $\epsilon(x_0) = 0$ and $\epsilon'(x_0) = 0$. Then from (9):

$$\epsilon_1(x_0)\ h_1 + \epsilon_2(x_0)\ h_2 = 0$$

$$\epsilon_1'(x_0)\ h_1 + \epsilon_2'(x_0)\ h_2 = 0$$

or $h_1 = 0$, $h_2 = 0$ since Wronski's determinant of the fundamental system $\epsilon_1(x)$, $\epsilon_2(x)$ cannot be zero.

The expression (9) thus reduces to

$$\begin{aligned}\epsilon(x) = \frac{2\,C_{p+1}}{\sum_{\mu=-1}^{m} \mu^2\, B_{k-\mu}}\, h^{p-1}&\left\{\epsilon_2(x) \int_{x_0}^{x} \frac{\epsilon_1(x)}{\epsilon_1(x)\ \epsilon_2'(x) - \epsilon_1'(x)\ \epsilon_2(x)}\ \overset{(p+1)}{y}(x)\ dx\right.\\ &\left. - \epsilon_1(x) \int_{x_0}^{x} \frac{\epsilon_2(x)}{\epsilon_1(x)\ \epsilon_2'(x) - \epsilon_1'(x)\ \epsilon_2(x)}\ \overset{(p+1)}{y}(x)\ dx\right\}\end{aligned} \quad (10)$$

Since different interpolation formulas (2) of the same order p differ only in the values of the coefficients $B_{k-\mu}$, $B''_{k-\mu}$ and in the constant C_{p+1} resulting from these coefficients, it clearly follows from (10) that the ratio of the propagated truncation errors for two such different formulas (2) is equal to the ratio of the parameters

$$E = \frac{C_{p+1}}{\sum_{\mu=-1}^{m} \mu^2 B_{k-\mu}} \tag{11}$$

of these formulas.[b]

If not only the ratio of the errors but the errors themselves are wanted, formula (10) would have to be evaluated. But obviously only in very few exceptional cases can this be done, since in general no fundamental system $\epsilon_1(x)$, $\epsilon_2(x)$ is known and the (p+1)st derivative $y^{(p+1)}(x)$ of the solution of the differential equation is also unknown. In paragraph B we will evaluate (10) for such an exceptional case.

But in general, in order to compute the error $\epsilon(x)$ along with the computation of the solution $y(x)$, one will have to integrate numerically the differential equation (8) for this error. This can be done by simple extrapolation formulas such as:

$$\delta^2 \epsilon_k = h^2 \left\{ \left(\frac{\partial f}{\partial y}\right)_k \epsilon_k + 2E\, \delta^{p-1} f_k \right\} \tag{12}$$

Here the derivative $y^{(p+1)}(x) = f^{(p-1)}(x)$ in (8) has been replaced by the difference quotient $\delta^{p-1} f/h^{p-1}$, since the central difference $\delta^{p-1} f$ is easily obtainable from the difference scheme of the f-values.

Formula (12) can be considered as the first part of Milne's formula (18) or Gauss's formula (22); the truncation error of (12) reads $\frac{1}{12} y^{IV}(x_k) h^4$.

Whenever the round-off error can be neglected for the computation of $\epsilon(x)$, we have obtained from (12) reasonably close approximations for the total error. We refer to our examples in paragraph D.

B. ERROR PROPAGATION FOR A SPECIAL DIFFERENTIAL EQUATION

Let us consider the special differential equation:

$$y'' = f(x,y) = -y \tag{13}$$

b. The expression (11) is identical with formula (28) in NASA TN D-599 (footnote a) if differential equations (1) and interpolation formulas (2) are considered in the TN.

and the initial conditions:

$$x_0 = 0, \; y_0 = 1, \; y_0' = 0$$

The solution of this problem then reads:

$$y = \cos x \tag{14}$$

For this example, $\epsilon(x)$ can easily be computed from formula (10). For the evaluation of (10), let us assume $p = 7$, as we will always do in our later examples in Sections II. D and III. D.

Then:

$$\overset{(p+1)}{y}(x) = \cos x$$

As a fundamental system of the homogeneous equation (8), the functions

$$\epsilon_1(x) = \cos x, \; \epsilon_2(x) = \sin x$$

can be used.

Formula (10) then reads:

$$\epsilon(x) = \frac{2\, C_8}{\sum\limits_{\mu=-1}^{m} \mu^2 B_{k-\mu}} h^6 \left\{ \sin x \int_0^x \cos^2 x \, dx - \cos x \int_0^x \cos x \sin x \, dx \right\}$$

or:

$$\epsilon(x) = \frac{C_8}{\sum\limits_{\mu=-1}^{m} \mu^2 B_{k-\mu}} h^6 \, x \sin x \tag{15}$$ [c]

Note that the error (15) does not again have the character of a cosine or sine wave, but is of the form $x \sin x$. This means that the increase of $\epsilon(x)$ with increasing x can only

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c. If $\epsilon(x_0) = 0$, $\epsilon'(x_0) = 0$ is replaced by $\epsilon(x_0) = 0$, $\epsilon(x_0 - h) = 0$, formula (15) changes only slightly to:

$$\epsilon(x) = \frac{C_8}{\sum\limits_{\mu=-1}^{m} \mu^2 B_{k-\mu}} h^6 \, (x + h) \sin x \tag{15a}$$

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be delayed but not prevented by a small factor $C_8 \Big/ \sum_{\mu=-1}^{m} \mu^2 B_{k-\mu}$.

C. ERROR PROPAGATION FOR THE INTERPOLATION FORMULAS OF MILNE AND GAUSS

Here and in Section III. C. we only consider interpolation formulas which are based on differences of the f-values, since for such formulas the difference $\delta^{p-1} f_k$ that enters the error equation (12) can easily be obtained from the difference scheme of the f-values.

1. The Backward-Difference Formula of Milne.[d] We replace the right-hand side of (1) by Newton's interpolation formula with ascending differences:

$$f = \sum_{\rho=0}^{p-2} \binom{u+\rho-2}{\rho} \nabla^\rho f_{k+1} \tag{16}$$

where we have put: $u = \frac{1}{h}(x-x_k)$.

Integrating (16) from $x = x_{k-1}$ to x or from $u = -1$ to u, yields

$$y' - y'_{k-1} = h \sum_{\rho=0}^{p-2} \nabla^\rho f_{k+1} \int_{-1}^{u} \binom{u+\rho-2}{\rho} du$$

Another integration with the same limits results in:

$$y - y_{k-1} - y'_{k-1}(x-x_{k-1}) = h^2 \sum_{\rho=0}^{p-2} \nabla^\rho f_{k+1} \int_{-1}^{u} \int_{-1}^{u} \binom{u+\rho-2}{\rho} du\, du \tag{17}$$

Writing (17) for $x = x_k$ ($u = 0$) and for $x = x_{k+1}$ ($u = 1$), and eliminating from these two equations the term with y'_{k-1}, leads to Milne's formula:

$$y_{k+1} - 2y_k + y_{k-1} = \delta^2 y_k = h^2 \sum_{\rho=0}^{p-2} M_\rho \nabla^\rho f_{k+1} \tag{18}$$

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d. Bennett, A. A., Milne, W. E., and Bateman, H.: Numerical Integration of Differential Equations. Dover Publications (New York), 1956, pp. 81-83.

with

$$M_\rho = \int_{-1}^{+1} \int_{-1}^{u} \binom{u+\rho-2}{\rho} du\, du - 2 \int_{-1}^{0} \int_{-1}^{u} \binom{u+\rho-2}{\rho} du\, du \tag{19}$$

For the first term of the truncation error of Milne's formula (18) we find:

$$T_k = M_{p-1}\, y^{(p+1)}(x_k)\, h^{p+1} \tag{20}$$

The first coefficients M_ρ are presented in Table I.

TABLE I: VALUES OF M_ρ

ρ	0	1	2	3	4
M_ρ	1	-1	1/12	0	-1/240
ρ	5	6	7	8	
M_ρ	-1/240	-221/60480	-19/6048	-9829/3628800	

For the error parameter 2E of equation (12), we find in the case of Milne's formula:

$$2E = -M_{p-1} \tag{21}$$

2. <u>The Central-Difference Formula of Gauss</u>.[e] The central-difference formula of Gauss reads:

$$y_{k+1} = h^2 \left\{ \delta^{-2} f_{k+1} + \sum_{2\rho=0}^{p-5} N_{2\rho+2}\, \delta^{2\rho} f_{k+1} \right\} \tag{22}$$

with $p \geqq 5$ and odd.

The first coefficients $N_{2\rho+2}$ of (22) are given in Table II.

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e. For the derivation of this formula see, for instance, Nyström, E. J., Acta Societatis Scientiarum Fennicae 50 (1925), No. 13, pages 33-37.

TABLE II: VALUES OF $N_{2\rho+2}$

ρ	0	1	2	3
$N_{2\rho+2}$	1/12	-1/240	31/60480	-289/3628800

According to the definition of the second sum function $\delta^{-2} f$ we have

$$\delta^{-2} f_{k+1} - 2\, \delta^{-2} f_k + \delta^{-2} f_{k-1} = f_k \tag{23}$$

and according to the definition of the $(2\rho+2)$nd difference $\delta^{2\rho+2} f$:

$$\delta^{2\rho} f_{k+1} - 2\, \delta^{2\rho} f_k + \delta^{2\rho} f_{k-1} = \delta^{2\rho+2} f_k$$

From (22) we then obtain:

$$y_{k+1} = 2\, y_k - y_{k-1} + h^2 \left\{ f_k + \sum_{2\rho=0}^{p-5} N_{2\rho+2}\, \delta^{2\rho+2} f_k \right\} \tag{24}$$

The first term of the truncation error of (22) or (24) is found to be:

$$T_k = N_{p-1} \overset{(p+1)}{y} (x_k)\, h^{p+1} \tag{25}$$

and the error parameter 2E for (22) or (24) becomes:

$$2E = - N_{p-1} \tag{26}$$

From (21) and (26) and the values of Tables I and II, it follows that the propagated truncation error of Gauss's formula is, for example, for $p = 7$ only 14 per cent and for $p = 9$ only 3 per cent of the error of Milne's formula.

D. EXAMPLES

All examples presented in this paper were computed on an IBM-7090 machine in double precision (16 digits). For all examples we have computed 100,000 steps of integration in order to observe the error behavior over a longer period of time. For each example all steps were computed with a constant step size. To start the integration procedure we have used the first f-values as obtained from the exact solution of the problem.

Example A: $y'' = -y$, $x_0 = 0$, $y_0 = 1$, $y_0' = 0$, $h = 0.04$ Solution: $y = \cos x$

Example B: $y'' = 2/x^4\, y$, $x_0 = 1$, $y_0 = 1$, $y_0' = -1$, $h = 0.01$ Solution: $y = 1/x$

Example C: $y'' = -1/4xy$, $x_0 = 1$, $y_0 = 1$, $y_0' = -1/2$, $h = 0.02$ Solution: $y = \sqrt{x}$

In the following Tables IIIa and IIIb we present for these three examples the errors after 1, 50,000, and 100,000 steps of integration. Table IIIa shows the errors for Milne's formula (18) and Table IIIb for Gauss's formula (22). Both tables show, in the third column, the actual errors $y_{k+1} - y(x_{k+1})$, and in the fourth column the approximated errors obtained from (12). For example A we have also presented the error values from (15a).

TABLE IIIa. ERRORS FOR FORMULA (18)

Example	No of Steps	Actual Errors	Approximated Errors from (12)	from (15a)
A	1	$0.2393 \cdot 10^{-13}$	$0.2394 \cdot 10^{-13}$	$0.2394 \cdot 10^{-13}$
	50,000	$0.1416 \cdot 10^{-7}$	$0.1324 \cdot 10^{-7}$	$0.1392 \cdot 10^{-7}$
	100,000	$-0.1943 \cdot 10^{-7}$	$-0.2312 \cdot 10^{-7}$	$-0.2046 \cdot 10^{-7}$
B	1	$0.1621 \cdot 10^{-13}$	$0.1484 \cdot 10^{-13}$	
	50,000	$0.3654 \cdot 10^{-9}$	$0.3347 \cdot 10^{-9}$	
	100,000	$0.2304 \cdot 10^{-9}$	$0.2114 \cdot 10^{-9}$	
C	1	$-0.5906 \cdot 10^{-13}$	$-0.5129 \cdot 10^{-13}$	
	50,000	$-0.6827 \cdot 10^{-7}$	$-0.5980 \cdot 10^{-7}$	
	100,000	$-0.1577 \cdot 10^{-6}$	$-0.1380 \cdot 10^{-6}$	

TABLE IIIb. ERRORS FOR FORMULA (22)

Example	No of Steps	Actual Errors	Approximated Errors from (12)	Approximated Errors from (15a)
A	1	$-0.3775 \cdot 10^{-14}$	$-0.3358 \cdot 10^{-14}$	$-0.3358 \cdot 10^{-14}$
	50,000	$-0.1951 \cdot 10^{-8}$	$-0.1895 \cdot 10^{-8}$	$-0.1953 \cdot 10^{-8}$
	100,000	$0.2875 \cdot 10^{-8}$	$0.3243 \cdot 10^{-8}$	$0.2870 \cdot 10^{-8}$
B	1	$-0.2331 \cdot 10^{-14}$	$-0.2082 \cdot 10^{-14}$	
	50,000	$-0.4625 \cdot 10^{-10}$	$-0.4695 \cdot 10^{-10}$	
	100,000	$-0.2912 \cdot 10^{-10}$	$-0.2966 \cdot 10^{-10}$	
C	1	$0.7327 \cdot 10^{-14}$	$0.7194 \cdot 10^{-14}$	
	50,000	$0.7980 \cdot 10^{-8}$	$0.8388 \cdot 10^{-8}$	
	100,000	$0.1831 \cdot 10^{-7}$	$0.1935 \cdot 10^{-7}$	

It might be possible to obtain a better approximation of the actual errors in the third column by a more accurate treatment of the error propagation, but the improvement that can be gained is probably not worth the complications that arise in a more accurate theory. Since we only want to know the approximate magnitude of the error, the simple and fast procedure that we have described in paragraph A seems to be sufficient.

SECTION III. ERROR PROPAGATION OF INTERPOLATION FORMULAS FOR THE DIFFERENTIAL EQUATION $y'' = f(x, y, y')$

A. THEORY OF THE ERROR PROPAGATION

We now consider differential equations:

$$y'' = f(x, y, y') \tag{27}$$

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and interpolation formulas of the form:

$$\left.\begin{aligned} h\, y'_{k+1} &= \sum_{\mu=-1}^{m} A_{k-\mu}\, y_{k-\mu} + h^2 \sum_{\mu=-m'}^{m} A''_{k-\mu}\, f_{k-\mu} \\ y_{k+1} &= \sum_{\mu=0}^{m} B_{k-\mu}\, y_{k-\mu} + h^2 \sum_{\mu=-m'}^{m} B''_{k-\mu}\, f_{k-\mu} \end{aligned}\right\} \tag{28}$$

Expanding the first equation (28) in a Taylor series for $x = x_k$ and comparing the coefficients for the first h-terms of the expansion results in the following relations for the coefficients $A_{k-\mu}$, $A''_{k-\mu}$:

$$\left.\begin{aligned} \sum_{\mu=-1}^{m} A_{k-\mu} &= 0 \\ 1 + \sum_{\mu=-1}^{m} \mu\, A_{k-\mu} &= 0 \\ 1 - \frac{1}{2} \sum_{\mu=-1}^{m} \mu^2 A_{k-\mu} - \sum_{\mu=-m'}^{m} A''_{k-\mu} &= 0 \end{aligned}\right\} \tag{29}$$

For the coefficients $B_{k-\mu}$, $B''_{k-\mu}$ the equations (3) still hold.

By introducing the errors:

$$\left.\begin{aligned} y_{k-\mu} - y(x_{k-\mu}) &= \epsilon_{k-\mu} \\ y'_{k-\mu} - y'(x_{k-\mu}) &= \eta_{k-\mu} \end{aligned}\right\} \tag{30}$$

in (28) we obtain for these errors:

$$\left.\begin{aligned} -h\, \eta_{k+1} + \sum_{\mu=-1}^{m} A_{k-\mu}\, \epsilon_{k-\mu} + h^2 \sum_{\mu=-m'}^{m} A''_{k-\mu} \left\{ \left(\frac{\partial f}{\partial y}\right)_{k-\mu} \epsilon_{k-\mu} + \left(\frac{\partial f}{\partial y'}\right)_{k-\mu} \eta_{k-\mu} \right\} \\ + \ldots = h\, T'_k \\ \sum_{\mu=-1}^{m} B_{k-\mu}\, \epsilon_{k-\mu} + h^2 \sum_{\mu=-m'}^{m} B''_{k-\mu} \left\{ \left(\frac{\partial f}{\partial y}\right)_{k-\mu} \epsilon_{k-\mu} + \left(\frac{\partial f}{\partial y'}\right)_{k-\mu} \eta_{k-\mu} \right\} + \ldots = T_k \end{aligned}\right\} \tag{31}$$

denoting the truncation errors for $x = x_k$ by T'_k and T_k, respectively. In the following we always assume that the truncation errors $h\,T'_k$ and T_k are of the same order in h.[f] Then we can write:

$$\left.\begin{aligned} T_k &= C_{p+1}\, y^{(p+1)}(x_k)\, h^{p+1} \\ h\, T'_k &= C'_p\, y^{(p+1)}(x_k)\, h^{p+1} \end{aligned}\right\} \qquad (32)$$

We now introduce the Taylor expansions:

$$\left.\begin{aligned} \epsilon_{k-\mu} &= \epsilon_k - \mu h\, \epsilon'_k + \frac{1}{2}\mu^2 h^2\, \epsilon''_k - \dots + \dots \\ \eta_{k-\mu} &= \eta_k - \mu h\, \eta'_k + \frac{1}{2}\mu^2 h^2\, \eta''_k - \dots + \dots \end{aligned}\right\} \qquad (33)$$

in (31). By making use of (3) and (29) and by neglecting higher order terms in the same way as we did in Section II. A., we find for the propagated truncation error the following differential equations:

$$\left.\begin{aligned} \epsilon' - \eta &= y^{(p+1)}(x)\, h^p \left\{ C'_p - C_{p+1} \frac{\sum\limits_{\mu=-1}^{m} \mu^2 A_{k-\mu}}{\sum\limits_{\mu=-1}^{m} \mu^2 B_{k-\mu}} \right\} \\ \epsilon'' - \frac{\partial f}{\partial y}\epsilon - \frac{\partial f}{\partial y'}\eta &= \frac{2\, C_{p+1}}{\sum\limits_{\mu=-1}^{m} \mu^2 B_{k-\mu}}\, y^{(p+1)}(x)\, h^{p-1} \end{aligned}\right\} \qquad (34)$$

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f. Interpolation formulas for which T'_k is already of the same order in h as T_k do not improve the error propagation compared with formulas for which (32) holds. We therefore restrict ourselves to formulas for which (32) holds.

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We solve the first equation (34) with respect to η and substitute this η-value in the second equation (34). Thus we find:

$$\epsilon'' - \frac{\partial f}{\partial y'} \epsilon' - \frac{\partial f}{\partial y} \epsilon = \frac{2 C_{p+1}}{\sum_{\mu=-1}^{m} \mu^2 B_{k-\mu}} y^{(p+1)}(x) \, h^{p-1} \tag{35}$$

In (35) we have neglected the term with $y^{(p+1)}(x) \, h^p$, retaining on the right hand of (35) only the term with the lowest h-power.

Comparing (35) with (8) we see that these two formulas differ only by the term $\frac{\partial f}{\partial y'} \epsilon'$ on the left-hand side. This means that the solution (10) of (18) also holds for (35), provided that $\epsilon_1(x)$, $\epsilon_2(x)$ is now a fundamental system of the homogeneous equation (35).

According to (10), $\epsilon(x)$ as well as $\epsilon'(x)$ contains the factor h^{p-1}. On the other hand, according to the first equation (34), $\eta(x)$ and $\epsilon'(x)$ differ by a term that contains the power h^p. This means, because of the terms we have neglected, that we can identify $\eta(x)$ with $\epsilon'(x)$.

Then in the case of the differential equation (27) and the interpolation formulas (28), the propagated truncation errors $\epsilon(x)$ and $\eta(x) = \epsilon'(x)$ are determined as solutions of (35).

It is evident that because of the terms we have neglected, the truncation error T'_k of the formula for y'_{k+1} nowhere enters the formulas for the error propagation.

Since the right-hand sides of (35) and (8) are identical, the ratio of the propagated truncation errors $\epsilon(x)$ and $\eta(x)$ for two different formulas (28) is again identical with the ratio of the E-values (11) for these two formulas.

This suggests the idea: (a) to retain also in the case of the differential equation (27) formulas for y_{k+1} which have been recognized as favorable with respect to error propagation in the case of the differential equation (1) - we mean such formulas as Milne's formula (18) or Gauss's formula (22); and (b) to establish for y'_{k+1} formulas of the form of the first equation (28) with the only restriction being that the truncation errors T_k and $h\,T'_k$ - according to (32) - must be of the same order in h. We shall present such formulas for y'_{k+1} in paragraph C.

Applying interpolation formulas of the form (28), the E-values that we have stated in Section II for differential equations (1) and interpolation formulas (2) will also hold in the more general case of differential equations (27).

This is, of course, only possible for interpolation formulas of the form (28). For if we consider as an example interpolation formulas of the Adams type:

$$\left.\begin{aligned} y'_{k+1} &= y'_k + h \sum_{\mu} A''_{k-\mu} f_{k-\mu} \\ y_{k+1} &= y_k + y'_k h + h^2 \sum_{\mu} B''_{k-\mu} f_{k-\mu} \end{aligned}\right\} \tag{36}$$

we obtain, instead of (34):

$$\left.\begin{aligned} \epsilon' - \eta &= - C_{p+1} \overset{(p+1)}{y} (x) \; h^p \\ \eta' - \frac{\partial f}{\partial y} \epsilon - \frac{\partial f}{\partial y'} \eta &= - C'_p \overset{(p+1)}{y} (x) \; h^{p-1} \end{aligned}\right\} \tag{37}$$

In (34) the lowest power of h, h^{p-1}, is multiplied by the factor C_{p+1} (for the formulas of Milne and Gauss in Section II we have $\sum_{m=-1}^{m} \mu^2 B_{k-\mu} = -2$). In (37), however, the power h^{p-1} carries the factor C'_p. But for all known interpolation formulas the coefficient C_{p+1} in (32) is considerably smaller than C'_p. For this reason our interpolation formulas (28) are more appropriate than, for example, Adams type formulas, if a favorable error propagation is desired.

For the computation of the errors $\epsilon(x)$ and $\eta(x) = \epsilon'(x)$, we again have to integrate the differential equation (35) numerically. We again apply very simple extrapolation formulas; namely:

$$\left.\begin{aligned} \delta^2 \left(\frac{\epsilon}{h}\right)_k &= h \left\{ \left(\frac{\partial f}{\partial y}\right)_k \epsilon_k + \left(\frac{\partial f}{\partial y'}\right)_k \epsilon'_k + 2\,E\,\delta^{p-1} f_k \right\} = h\,F_k \\ \epsilon'_{k+1} &= \delta \left(\frac{\epsilon}{h}\right)_{k+1/2} + h \left\{ \frac{5}{6} F_k - \frac{1}{3} F_{k-1} \right\} \end{aligned}\right\} \tag{38}$$

The first formula (38) corresponds to formula (12) of Section II; the second formula (38), which can easily be derived, has the truncation error:

$\frac{7}{24} y^{IV}(x)\, h^3.$

The examples in paragraph D below will show that (38) leads to satisfactory approximations for the propagated truncation error.

B. ERROR PROPAGATION FOR A SPECIAL DIFFERENTIAL EQUATION

We again consider a special differential equation for which the solution (10) of the error equation (35) can be computed directly:

$$y'' = f(x, y, y') = -y - y' - \sin x\ , \qquad (39)$$

For the initial conditions

$$x_0 = 0,\ y_0 = 1,\ y'_0 = 0$$

the differential equation (39) has the solution:

$$y = \cos x \qquad (40)$$

For p = 7 we have again:

$$y^{(p+1)}(x) = \cos x$$

As a fundamental system of the homogeneous equation (35) we obtain:

$$\epsilon_1(x) = e^{-\frac{1}{2}x} \cos\left(\tfrac{1}{2}\sqrt{3}\,x\right), \qquad \epsilon_2(x) = e^{-\frac{1}{2}x} \sin\left(\tfrac{1}{2}\sqrt{3}\,x\right)$$

For our problem (39), (40), the general solution of the non-homogeneous equation (35) is found to be:

$$\epsilon(x) = \frac{2}{3}\sqrt{3}\ \frac{C_8}{\sum\limits_{\mu=-1}^{m} \mu^2 B_{k-\mu}}\ h^6 \left\{\sqrt{3}\ \sin x - 2h_1 e^{-\frac{1}{2}x} \cos\left(\tfrac{1}{2}\sqrt{3}\,x\right) + 2h_2 e^{-\frac{1}{2}x} \sin\left(\tfrac{1}{2}\sqrt{3}\,x\right)\right\},$$

h_1, h_2 being constants, the values of which are determined by the initial values for $\epsilon(x_0)$ and $\epsilon'(x_0)$ or for $\epsilon(x_0)$ and $\epsilon(x_0 - h)$.

Whatever the values for h_1 and h_2 may be, due to the factor $e^{-\frac{1}{2}x}$ the terms with h_1 and h_2 can be neglected for sufficiently large values of x. For large values of x the errors $\epsilon(x)$ and $\epsilon'(x)$ then reduce to :

$$\epsilon(x) = \frac{2\,C_8}{\sum\limits_{\mu=-1}^{m} \mu^2\, B_{k-\mu}}\, h^6 \sin x; \quad \epsilon'(x) = \eta(x) = \frac{2\,C_8}{\sum\limits_{\mu=-1}^{m} \mu^2\, B_{k-\mu}}\, h^6 \cos x \tag{41}$$

Contrary to our former result (15), the error $\epsilon(x)$ has here the character of a sine-wave and does not grow with x. [g]

C. FORMULAS FOR y'_{k+1} SUPPLEMENTING THE FORMULAS OF MILNE AND GAUSS IN THE CASE OF A DIFFERENTIAL EQUATION $y'' = f(x,y,y')$

1. <u>A Supplementary Formula for Milne's Formula.</u> We start from the interpolation formulas of Adams:

$$\left.\begin{aligned} y'_{k+1} &= y'_k + h \sum_{\rho=0}^{p-2} D'_\rho\, \nabla^\rho f_{k+1} \\ y_{k+1} &= y_k + y'_k\, h + h^2 \sum_{\rho=0}^{p-2} D_\rho\, \nabla^\rho f_{k+1} \end{aligned}\right\} \tag{42}$$

the coefficients of which are given by:

$$\left.\begin{aligned} D'_\rho &= \int_{-1}^{0} \binom{u+\rho-1}{\rho} du \\ D_\rho &= \int_{-1}^{0} \int_{-1}^{u} \binom{u+\rho-1}{\rho} du\, du \end{aligned}\right\} \rho = 0,\ 1,\ 2,\ \ldots \tag{43}$$

---

g. The fact that the error propagation depends on the fundamental system of the homogeneous equation (35) or, in the case of Section II, equation (8), is well-known and needs no further discussion, Solving the second formula (42) with respect to y'_k and substituting this expression for y'_k in the first equation (42) leads to:

$$h\, y'_{k+1} = y_{k+1} - y_k + h^2 \sum_{\rho=0}^{p-2} M'_\rho \nabla^\rho f_{k+1} \tag{44}$$

with:

$$M'_\rho = D'_\rho - D_\rho \tag{45}$$

Formula (44) has the form of the first interpolation formula (28) and can be used in combination with Milne's formula (18).

Table IV presents the first coeefficients M'_ρ of (44).

TABLE IV. VALUES OF M'_ρ

ρ	0	1	2	3	4
M'_ρ	1/2	-1/6	-1/24	-1/45	-7/480
ρ	5	6	7	8	
M'_ρ	-107/10080	-199/24192	-6031/907200	-5741/1036800	

The first term of the truncation error of (44) becomes:

$$h\, T'_k = M'_{p-1}\, y^{(p+1)}(x_k)\, h^{p+1} \tag{46}$$

As explained in paragraph A, the pair of formulas (18), (44) leads in the case of the differential equation $y'' = f(x,y,y')$ to the same small error parameter E as formula (18) does in the case of the differential equation $y'' = f(x,y)$.

2. <u>A Supplementary Formula for Gauss's Formula.</u> In the case of the differential equation $y'' = f(x,y,y')$, Gauss has already combined the central difference formula (22) with the following central difference formula for y'_{k+1}:

$$y'_{k+1} = h \left\{ \delta^{-1} f_{k+1/2} + \frac{1}{2} f_{k+1} + \sum_{2\rho=0}^{p-5} N'_{2\rho+2}\, \overline{\delta}^{2\rho+1} f_{k+1} \right\} \quad (p \geq 5 \text{ and odd}) \tag{47}$$

The first coefficients $N'_{2\rho+2}$ of (47) are listed in Table V.

TABLE V. VALUES OF $N'_{2\rho+2}$

ρ	0	1	2	3
$N'_{2\rho+2}$	-1/12	11/720	-191/60480	2497/3628800

However, the combination of the formulas (22) and (47) meets with certain difficulties with respect to the propagation of the truncation error. These difficulties, which we will now explain, are related to the build-up of the sum functions $\delta^{-1} f_{k+\frac{1}{2}}$ in (47) and $\delta^{-2} f_k$ in (22).

Applying (22) to the integration of a differential equation (1), the build-up of the second sum function is given by (23). To start the integration the first two values of the second sum function:

$$\left.\begin{aligned} \delta^{-2} f_{k-1} &= \frac{y_{k-1}}{h^2} - \sum_{2\rho=0}^{p-5} N_{2\rho+2}\, \delta^{2\rho} f_{k-1} \\ \delta^{-2} f_k &= \frac{y_k}{h^2} - \sum_{2\rho=0}^{p-5} N_{2\rho+2}\, \delta^{2\rho} f_k \end{aligned}\right\} \tag{48}$$

have to be determined from the initial difference scheme of the f-values. According to (23), the next value $\delta^{-2} f_{k+1}$ is then based on the two values (48) and on f_k, etc.

If we now apply (22) and (47) to the integration of a differential equation (27) for each of the two sum functions $\delta^{-1} f_{k+\frac{1}{2}}$ and $\delta^{-2} f_{k+1}$, one starting value:

$$\left.\begin{aligned} \delta^{-1} f_{k-1/2} &= \frac{y'_k}{h} - \frac{1}{2} f_k - \sum_{2\rho=0}^{p-5} N'_{2\rho+2}\, \overline{\delta}^{2\rho+1} f_k \\ \delta^{-2} f_k &= \frac{y_k}{h^2} - \sum_{2\rho=0}^{p-5} N_{2\rho+2}\, \delta^{2\rho} f_k \end{aligned}\right\} \tag{49}$$

has to be determined from the initial difference scheme of the f-values. The succeeding values $\delta^{-1} f_{k+\frac{1}{2}}$ and $\delta^{-2} f_{k+1}$ are then computed from:

$$
\left.\begin{aligned}
\delta^{-1} f_{k+\frac{1}{2}} &= \delta^{-1} f_{k-\frac{1}{2}} + f_k \\
\delta^{-2} f_{k+1} &= \delta^{-2} f_k + \delta^{-1} f_{k+\frac{1}{2}} = \delta^{-2} f_k + \delta^{-1} f_{k-\frac{1}{2}} + f_k
\end{aligned}\right\} \tag{50}
$$

etc.

If we now introduce the first equation (50) and the first equation (49) in equation (47), we find:

$$
y'_{k+1} = y'_k + h \left\{ \frac{1}{2} \left(f_k + f_{k+1} \right) + \sum_{2\rho=0}^{p-5} N'_{2\rho+2} \, \bar{\delta}^{2\rho+2} \, f_{k+\frac{1}{2}} \right\} \tag{51}
$$

In the same way the introduction of (50) and (49) in (22) leads to:

$$
y_{k+1} = y_k + y'_k h + h^2 \left\{ \frac{1}{2} f_k + \sum_{2\rho=0}^{p-5} N_{2\rho+2} \, \delta^{2\rho+1} \, f_{k+\frac{1}{2}} - \sum_{2\rho=0}^{p-5} N'_{2\rho+2} \, \bar{\delta}^{2\rho+1} \, f_k \right\} \tag{52}
$$

The first term of the truncation error of formulas (51) and (52) reads:

$$
h \, T'_k = N'_{p-1} \, y^{(p+1)}(x_k) \, h^{p+1} \tag{53}
$$

and

$$
T_k = \left(N_{p-1} - N'_{p-1} \right) y^{(p)}(x_k) \, h^p, \tag{54}
$$

respectively.

We have thus obtained the result that Gauss's formulas (22), (47) become equivalent to (51), (52), this means to formulas of the type (36), if we build up the sum functions according to (49) and (50). But we have already seen that formulas of the type (36) are inferior to formulas of the type (28) as far as the propagation of the truncation error is concerned. Furthermore, formula (52) does not have the required accuracy, since the truncation error (54) is proportional to $y^{(p)}(x_k)\, h^p$ instead of proportional to $y^{(p+1)}(x_k)\, h^{p+1}$.

Let us now try to transform Gauss's formulas (22), (47) into formulas of the type (28) by a different build-up of the sum functions. This can be done if we base the sum functions not on the starting values (49), but on the starting values (48), as we have done in the case of the differential equation (1), and compute the next values of the sum functions from:

$$\left.\begin{aligned} \delta^{-2} f_{k+1} &= 2\,\delta^{-2} f_k - \delta^{-2} f_{k-1} + f_k \\ \delta^{-1} f_{k+\frac{1}{2}} &= \delta^{-2} f_{k+1} - \delta^{-2} f_k \end{aligned}\right\} \tag{55}$$

If we do so, (22) again transforms to (24) whereas (47) becomes:

$$y'_{k+1} = \frac{1}{h}(y_{k+1} - y_k) + h\left\{\frac{1}{2} f_{k+1} + \sum_{2\rho=0}^{p-5} N'_{2\rho+2}\,\bar{\delta}^{2\rho+1} f_{k+1} - \sum_{2\rho=0}^{p-5} N_{2\rho+2}\,\delta^{2\rho+1} f_{k+\frac{1}{2}}\right\} \tag{56}$$

Formula (56) has the desired form of the first interpolation formula (28). However, we meet here again with the difficulty that the truncation error of (56) is not of the required order in h, since we find for the first term of this truncation error:

$$h\,T'_k = (N'_{p-1} - N_{p-1})\,\overset{(p)}{y}(x_k)\,h^p \tag{57}$$

As in (54), we are again short in (57) by one power of h.

To overcome this difficulty we now introduce in (47) an additional term that does not change the form of (56), but yields a truncation error of the required order as stated in the second formula (32). Such a modified formula (47), the sum functions of which are based on (48) and (55), combined with the unchanged formula (22) will then lead to the same favorable error parameter E that we have obtained for the formula (22) in the case of the differential equation $y'' = f(x,y)$.

As such a modified formula (47) we use:

$$y'_{k+1} = h\left\{\delta^{-1} f_{k+\frac{1}{2}} + \frac{1}{2} f_{k+1} + \sum_{2\rho=0}^{p-5} N'_{2\rho+2}\,\bar{\delta}^{2\rho+1} f_{k+1} + (N'_{p-1} - N_{p-1})\,\bar{\delta}^{p-2} f_{k+1}\right\} \tag{58}$$

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It can easily be verified that (58) has the desired properties.

The first term of the truncation error of (58) is found to be:

$$h\, T'_k = \frac{1}{2} N_{p-1}\, y^{(p+1)}(x_k)\, h^{p+1} \tag{59}$$

In the case of the formulas (22), (58) we can again compute the errors $\epsilon(x)$ and $\eta(x) = \epsilon'(x)$, along with the integration, by applying the simple extrapolation formulas (38).

D. EXAMPLES

Tables VIa and VIb report the error results of the supplemented Milne formula (18), (44) and for the modified Gauss formulas (22), (58) - sum functions built up according to (48) and (55) - for the following three examples:

Example D: $y'' = -y - y' - \sin x$; $x_0 = 0$, $y_0 = 1$, $y'_0 = 0$, $h = 0.04$, Solution: $y = \cos x$

Example E: $y'' = y' \cos x - y \sin x$; $x_0 = 0$, $y_0 = 1$, $y'_0 = 1$, $h = 0.02$, Solution: $y = e^{\sin x}$

Example F: $y'' = 2y'^2/y$, $x_0 = 1$, $y_0 = 1$, $y'_0 = -1$, $h = 0.02$, Solution: $y = 1/x$

In order to demonstrate that our modified Gauss Formulas (22), (58) indeed have smaller propagated truncation errors than the original Gauss Formulas (22), (47), we show in Table VII the error results for Example E using these two methods. The values of Table VII are the actual errors $y_{k+1} - y(x_{k+1})$. The difference in the magnitude of the errors is rather obvious for these two methods.

SECTION IV. FINAL REMARKS

We have already mentioned in the introduction that our statements about the error propagation only hold if the truncation error outweighs the round off error. Whether or not this is true for a certain step size of integration can easily be checked.

After the first step of integration, it is immediately seen from (12) and from (38), respectively, whether for the chosen step size the error ϵ still has the same order as the round-off error. We then select an initial step size for which the truncation error becomes dominant. This, of course, does not yet guarantee that the truncation error still remains dominant after a large number of integration steps. Therefore, it is advisable to inspect, along with the computation, the difference $\delta^{p-1} f_k$ that enters the formulas (12) and (38), respectively. If this difference starts to show irregularities, the round-off error has already affected the last columns of the difference scheme for the f-values.

However, in such a case, the round-off error has not yet necessarily affected our error values ϵ and ϵ', respectively. As long as the term $2E\,\delta^{p-1} f_k$ in (12) or (38) remains small compared with $\left(\frac{\partial f}{\partial y}\right)_k \epsilon_k$ or with $\left(\frac{\partial f}{\partial y}\right)_k \epsilon_k + \left(\frac{\partial f}{\partial y'}\right)_k \epsilon'_k$, our error values will not be materially influenced by the round-off error.

In Table VIII we present, for example C, the last steps which we have computed with Gauss's formula (22). The irregularities in $2E\,\delta^6 f_k$ are quite obvious in this table. They occur after 10,000 steps in much the same way as they do in this table after 100,000 steps. Nevertheless, our error values (12) are still in good agreement with the actual errors due to the fact that in this example (for the step size $h = 0.02$) the term $2E\,\delta^6 f_k$ always remains small compared with $\left(\frac{\partial f}{\partial y}\right)_k \epsilon_k$.

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TABLE VIa. ERRORS FOR FORMULAS (18), (44)

Ex.	No of Steps	Actual Errors	$\epsilon(x)$ Approximated Errors from (38)	from (41)	Actual Errors	$\epsilon'(x) = \eta(x)$ Approximated Errors from (38)	from (41)
D	1	$0.2354 \cdot 10^{-13}$	$0.2393 \cdot 10^{-13}$		$0.1920 \cdot 10^{-11}$	$0.1097 \cdot 10^{-11}$	
	50,000	$0.1323 \cdot 10^{-10}$	$0.1391 \cdot 10^{-10}$	$0.1392 \cdot 10^{-10}$	$-0.4899 \cdot 10^{-11}$	$-0.5499 \cdot 10^{-11}$	$-0.5500 \cdot 10^{-11}$
	100,000	$-0.9273 \cdot 10^{-11}$	$-0.1023 \cdot 10^{-10}$	$-0.1023 \cdot 10^{-10}$	$-0.1154 \cdot 10^{-10}$	$-0.1092 \cdot 10^{-10}$	$-0.1093 \cdot 10^{-10}$
E	1	$0.1998 \cdot 10^{-13}$	$0.2036 \cdot 10^{-13}$		$0.3280 \cdot 10^{-11}$	$0.1867 \cdot 10^{-11}$	
	50,000	$-0.3490 \cdot 10^{-7}$	$-0.3677 \cdot 10^{-7}$		$-0.1969 \cdot 10^{-7}$	$-0.2074 \cdot 10^{-7}$	
	100,000	$-0.7724 \cdot 10^{-7}$	$-0.8305 \cdot 10^{-7}$		$0.2843 \cdot 10^{-7}$	$0.3057 \cdot 10^{-7}$	
F	1	$0.4588 \cdot 10^{-11}$	$0.4414 \cdot 10^{-11}$		$0.7464 \cdot 10^{-9}$	$0.4046 \cdot 10^{-9}$	
	50,000	$0.1748 \cdot 10^{-11}$	$0.1848 \cdot 10^{-11}$		$-0.1744 \cdot 10^{-14}$	$-0.1844 \cdot 10^{-14}$	
	100,000	$0.8732 \cdot 10^{-12}$	$0.9249 \cdot 10^{-12}$		$-0.4360 \cdot 10^{-15}$	$-0.4619 \cdot 10^{-15}$	

TABLE VIb. ERRORS FOR FORMULAS (22), (58)

Ex.	No of Steps	Actual Errors	$\epsilon(x)$ Approximated Errors from (38)	$\epsilon(x)$ Approximated Errors from (41)	Actual Errors	$\epsilon'(x) = \eta(x)$ Approximated Errors from (38)	$\epsilon'(x) = \eta(x)$ Approximated Errors from (41)
D	1	$-0.3775 \cdot 10^{-14}$	$-0.3359 \cdot 10^{-14}$		$-0.1233 \cdot 10^{-12}$	$-0.1539 \cdot 10^{-12}$	
	50,000	$-0.1870 \cdot 10^{-11}$	$-0.1952 \cdot 10^{-11}$	$-0.1953 \cdot 10^{-11}$	$0.7358 \cdot 10^{-12}$	$0.7731 \cdot 10^{-12}$	$0.7715 \cdot 10^{-12}$
	100,000	$0.1234 \cdot 10^{-11}$	$0.1435 \cdot 10^{-11}$	$0.1435 \cdot 10^{-11}$	$0.1367 \cdot 10^{-11}$	$0.1532 \cdot 10^{-11}$	$0.1533 \cdot 10^{-11}$
E	1	$-0.3109 \cdot 10^{-14}$	$-0.2849 \cdot 10^{-14}$		$-0.2123 \cdot 10^{-12}$	$-0.2611 \cdot 10^{-12}$	
	50,000	$0.4662 \cdot 10^{-8}$	$0.5149 \cdot 10^{-8}$		$0.2629 \cdot 10^{-8}$	$0.2904 \cdot 10^{-8}$	
	100,000	$0.1060 \cdot 10^{-7}$	$0.1163 \cdot 10^{-7}$		$-0.3903 \cdot 10^{-8}$	$-0.4281 \cdot 10^{-8}$	
F	1	$-0.5923 \cdot 10^{-12}$	$-0.5641 \cdot 10^{-12}$		$-0.4338 \cdot 10^{-10}$	$-0.5171 \cdot 10^{-10}$	
	50,000	$-0.2367 \cdot 10^{-12}$	$-0.2437 \cdot 10^{-12}$		$0.2363 \cdot 10^{-15}$	$0.2432 \cdot 10^{-15}$	
	100,000	$-0.1202 \cdot 10^{-12}$	$-0.1220 \cdot 10^{-12}$		$0.6006 \cdot 10^{-16}$	$0.6093 \cdot 10^{-16}$	

TABLE VII. EXAMPLE E: ERROR FOR FORMULAS (22), (58) AND (22), (47)

No of Steps	Error in y for (22), (58)	Error in y for (22), (47)	Error in y' for (22), (58)	Error in y' for (22), (47)
1	$-0.3109 \cdot 10^{-14}$	$-0.2649 \cdot 10^{-12}$	$-0.2123 \cdot 10^{-12}$	$0.8864 \cdot 10^{-12}$
20,000	$0.3897 \cdot 10^{-9}$	$-0.8224 \cdot 10^{-8}$	$-0.2032 \cdot 10^{-9}$	$0.4258 \cdot 10^{-8}$
40,000	$0.4212 \cdot 10^{-8}$	$-0.1603 \cdot 10^{-6}$	$-0.1894 \cdot 10^{-8}$	$0.7176 \cdot 10^{-7}$
60,000	$0.2165 \cdot 10^{-8}$	$-0.1274 \cdot 10^{-6}$	$0.2157 \cdot 10^{-8}$	$-0.1271 \cdot 10^{-6}$
80,000	$0.1510 \cdot 10^{-8}$	$-0.1071 \cdot 10^{-6}$	$-0.9018 \cdot 10^{-9}$	$0.6388 \cdot 10^{-7}$
100,000	$0.1060 \cdot 10^{-7}$	$-0.9271 \cdot 10^{-6}$	$-0.3903 \cdot 10^{-8}$	$0.3404 \cdot 10^{-6}$

TABLE VIII. EXAMPLE C: ERROR FOR FORMULA (22)

Number of Steps	$\left(\frac{\partial f}{\partial y}\right)_k \epsilon_k$	$2E\,\delta^6 f_k$	Error	
			Actual Error	Approximated Error from (12)
100,000	$0.120849 \cdot 10^{-14}$	$-0.174750 \cdot 10^{-22}$	$0.183110 \cdot 10^{-7}$	$0.193550 \cdot 10^{-7}$
100,001	$0.120848 \cdot 10^{-14}$	$+0.860723 \cdot 10^{-22}$	$0.183112 \cdot 10^{-7}$	$0.193552 \cdot 10^{-7}$
100,002	$0.120847 \cdot 10^{-14}$	$-0.132962 \cdot 10^{-21}$	$0.183114 \cdot 10^{-7}$	$0.193554 \cdot 10^{-7}$
100,003	$0.120846 \cdot 10^{-14}$	$+0.122867 \cdot 10^{-21}$	$0.183116 \cdot 10^{-7}$	$0.193557 \cdot 10^{-7}$
100,004	$0.120845 \cdot 10^{-14}$	$-0.741329 \cdot 10^{-22}$	$0.183118 \cdot 10^{-7}$	$0.193559 \cdot 10^{-7}$

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